REALIZABLE WARPED IIR FILTERS AND THEIR PROPERTIES

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ABSTRACT

Digital filters where unit delays are replaced with frequency-dependent delays, such as first order allpass sections, are often called warped filters since they implement filter specifications on a warped non-uniform frequency scale. Warped IIR (WIIR) filters cannot be realized directly due to delay-free loops. Specific solutions have been known that make WIIR filters realizable but no general approach has been available so far. In this paper we will explore the generation of such filters, including new filter structures. The robustness and computational efficiency of WIIR filters are studied and most potential applications are discussed.

1. INTRODUCTION

Several principles of warped digital signal processing have been published earlier. FFT on a warped frequency scale was first introduced by Oppenheim et al. [1] and warped linear prediction was published by Strube [2]. A recursive warped filter structure was introduced by Steiglitz [4]. Generalized methods using FAM functions have been developed by Laine et al. [3]. The idea of warped transversal filters has been systematically studied also under the concepts of Laguerre and Kautz filters; for good introductions see [5] and [6].

In addition to the warped FFT mentioned above, there have been some practical applications of warped filters such as modeling the body of the violin [7] and the guitar [9]. Recently we have applied the principles to auditory modeling [8], to HRTF filter implementation [11] and to the equalization of the loudspeaker response [10], as well as to audio coding using warped linear prediction [12].

Warped signal processing and digital filtering principles remain, however, widely unknown. Especially warped IIR filters have not been studied in detail although they reveal interesting potential for applications. In this paper we first present the idea of warped filters using an example of the warped FIR (WFIR) structure. Then, both previously known and new warped IIR (WIIR) structures are shown and analyzed.

2. BASICS OF WARPED FILTERS

The idea of warped filters is best illustrated using the FIR-like structures in Figure 1. If each unit delay of an FIR filter is replaced with a new delay element so that each new delay is frequency dependent (dispersive), the filter can be designed and realized on a warped frequency scale.
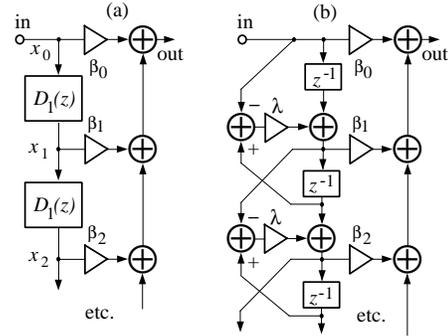
Figure 1: *The principle of warped filters as an FIR structure: a) with allpass delay elements and b) as a computationally efficient version.*

The design of warped filters may be based on any pair of functions of complex variable, $\tilde{z} = f(z)$ and $z = g(\tilde{z})$, so that functions $f(\cdot)$ and $g(\cdot)$ are one-to-one mappings of the unit disc onto itself, and $z = g\{f(z)\}$, i.e., they are inverse mappings. There exists only one rational function type that meets the requirement, the bilinear conformal mapping [13], which corresponds to the first order allpass filter

$$\tilde{z}^{-1} = D_1(z) = \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \tag{1}$$

where λ, $-1 < \lambda < 1$, is a warping parameter and $D_1(z)$ is a dispersive delay element. Figure 2 shows how the frequency warping characteristics depend on the parameter λ.

We may derive the design of the WFIR filter in Fig. 1 in the following way. The desired impulse response $h(n)$ and its z-transform $H(z)$ must be equal to the impulse response $\tilde{h}(k)$ and its z-transform $\tilde{H}(\tilde{z})$ in the warped domain, i.e.,

$$H(z) = \sum_{k=0}^{\infty} \tilde{h}(k)\,\tilde{z}^{-k} \quad and \quad \tilde{H}(\tilde{z}) = \sum_{n=0}^{\infty} h(n)\,z^{-n} \tag{2}$$

Mappings between sequences $h(n)$ and $\tilde{h}(k)$ are linear but not shift-invariant. The first form specifies the WFIR realization (= synthesis) structure yielding

$$H_{WFIR}(z) = \sum_{n=0}^{M} \tilde{h}(n)\,\tilde{z}^{-n} = \sum_{n=0}^{M} \beta_n \{D_1(z)\}^n \tag{3}$$

and the second form of (2) yields a method to compute the WFIR coefficients (= analysis). It is easy to show from (1) that both forms of (2) may be computed with the same
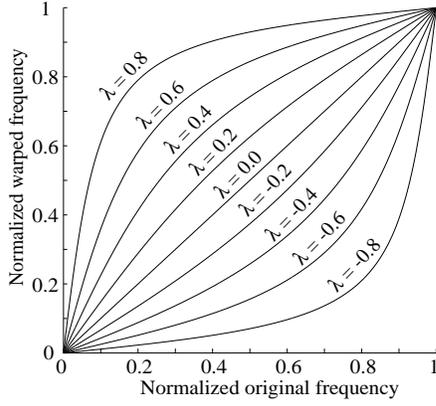
Figure 2: *Frequency warping characteristics of the first-order allpass section $D_1(z)$ for different values of λ. Frequencies are normalized to the Nyquist rate.*

Figure 3: *A non-realizable (direct-form II) WIIR structure from equation (3).*

warping structure but using coefficient λ for synthesis and $-\lambda$ for analysis.

Notice also that both forms of (2) yield responses of infinite length even if the sequence to be mapped is of finite length[1]. Since the coefficient sequence β_i must in practice be of finite length, we have to approximate $\check{h}(i)$, e.g., by truncation (3) or by windowing. The second form of (2), the 'prewarping' of the target response, can also be applied to the design of warped IIR filters below.

By expanding the bilinear mapping, inherent in the delays $D_1(z)$ of warped filters, we may—at least in theory—transform any warped filter into an equivalent traditional structure, such as direct form I or II. Warped implementations have, however, advantages that may compensate for the extra complexity of the warped delay elements:

- Filters can be designed directly on a warped frequency scale such as the psychoacoustically motivated Bark scale [14].

- Warped structures are more robust and require less precision if the poles and zeros are mapped so that they are more uniformly distributed over the warped frequency scale.

- The order of the warped filter may in some cases be considerably lower (e.g., 5–10 times in Bark scale modeling) than a filter designed on a uniform Hz scale.

- The warping parameter λ may be used as a control parameter for filters where the resonances and cutoff frequencies have to be controlled.

3. DIRECT FORM WIIR FILTERS

A general form for the transfer function of a warped IIR (WIIR) filter is

$$H_{WIIR}(z) = \frac{\sum_{i=0}^{M} \beta_i \left[D_1(z)\right]^i}{1 + \sum_{i=1}^{R} \alpha_i \left[D_1(z)\right]^i} \qquad (4)$$

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[1] Warped FIR filters have infinite impulse responses since all-pass elements are internally recursive. Thus the term WFIR is somewhat contradictory but describes well the structural analogy to transversal FIR filters.
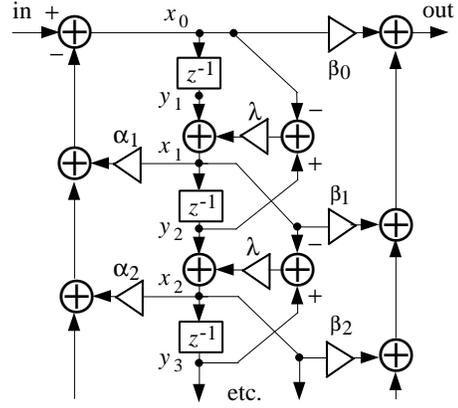
A realization problem appears since delay elements $D_1(z)$ contain a delay-free component as is seen from the form

$$D_1(z) = \frac{\left(1 - \lambda^2\right) z^{-1}}{1 - \lambda z^{-1}} - \lambda \qquad (5)$$

Direct implementation of (3), as shown in Fig. 3, is not possible because the feedback loops contain lag-free paths when $\lambda \neq 0$.

There exist several solutions that make WIIR filters realizable. Strube [2] proposed a structure where lowpass sections are used instead of allpass delays. This is shown in Fig. 4a for a warped 'all-pole' case. The allpass filter coefficients α_i (Fig. 3) are mapped to another set of coefficients γ_i for a realizable filter structure by formulas (9) and (10) in [2]. Unfortunately, this works in practice only for low-order filters with moderate warping, since a long sequence of lowpass sections makes the structure ill-conditioned. Problems arise as well when trying to expand Eq. (3) into a traditional IIR filter form. This is due to ill-conditioning of the bilinear mapping since it creates R zeros at $z = \lambda$, where R is the order of the filter.

Another realizable WIIR filter is possible using the structure given in Fig. 4b. It is formulated using delay elements that have highpass characteristics (for positive λ). The structure is made realizable by the technique that will be introduced below for WIIR filters with allpass elements. This structure suffers from similar drawbacks as the case of Fig. 4a.
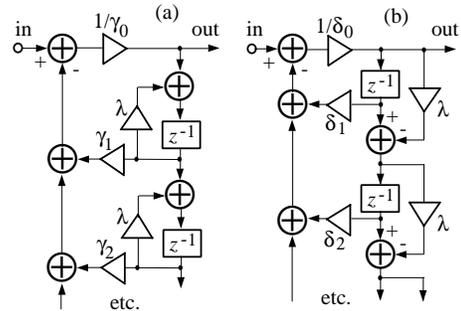
Figure 4: *Realizable WIIR all-pole filter (a) with lowpass elements and (b) with highpass elements.*
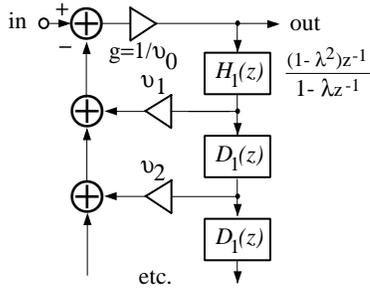
Figure 5: *WIIR all-pole structure proposed by Steiglitz.*

A more robust realizable WIIR structure that works also for high filter orders was proposed by Steiglitz [4]. An 'all-pole' version is shown in Fig. 5. (The feedforward part of a pole-zero filter is a straigthforward addition.) The delay chain preserves the allpass property except for the first step that has lowpass characteristics. A recursion formula for the computation of feedback coefficients v_i, given in [4], is exceptionally simple

$$
\begin{aligned}
v_R &= \alpha_R; \\
v_i &= \alpha_i - \lambda v_{i+1}; \qquad i = R-1, ..., 1 \qquad (6) \\
1/g &= v_0 = 1 - \lambda v_1;
\end{aligned}
$$

Now we introduce a new realizable WIIR structure where the delay chain has full all-pass characteristics. The first delay is a unit delay and the other ones are first-order allpass sections. The recursive feedbacks α_i of Fig. 3 are mapped to coefficients σ_i which feed back from the outputs of the unit delays of the allpass sections in order to avoid lag-free loops. A gain term $g = 1/\sigma_0$ is also needed. The feedforward part of the WIIR filter can be implemented directly, without modifications, using β_i coefficients. The resulting WIIR structure is shown in Fig. 6.

The computation of σ_i and g can be derived using notations of Fig. 3 as follows:

$$
x_0 = in - \sum_{i=1}^{R} \alpha_i x_i \quad and \quad x_i = y_i + \lambda (y_{i+1} - x_{i-1})
$$

Terms x_i may be eliminated by iterating downwards from index $i = R$, first writing

$$
x_0 = in - \sum_{i=1}^{R-1} \alpha_i x_i + \lambda \alpha_R x_{R-1} - \alpha_R y_R - \lambda \alpha_R y_{R+1}
$$

Using $S_R = \alpha_R$, $\sigma_{R+1} = \lambda S_R$, and $S_{i-1} = \alpha_{i-1} - \lambda S_i$

$$
x_0 = in - \sum_{i=1}^{R-2} \alpha_i x_i - S_{R-1} x_{R-1} - \alpha_R y_R - \sigma_{R+1} y_{R+1}
$$

Substitution of x_{R-1} and $\sigma_i = \lambda S_{i-1} + S_i$ yields

$$
x_0 = in - \sum_{i=1}^{R-3} \alpha_i x_i - S_{R-2} x_{R-2} - S_{R-1} y_{R-1} - \sum_{i=R}^{R+1} \sigma_i y_i
$$

The iteration is repeated until
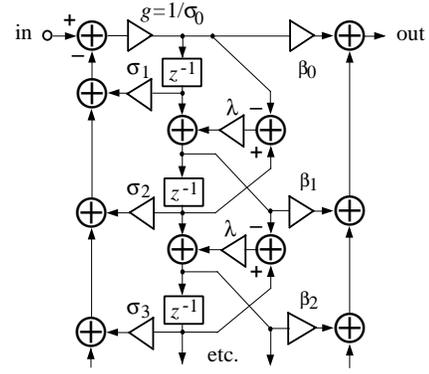
Figure 6: *A realizable WIIR structure with first-order all-pass delays and a single unit delay.*

$$
x_0 = in + \lambda S_1 x_0 - S_1 y_1 - \sum_{i=2}^{R+1} \sigma_i y_i
$$

and finally

$$
(1 - \lambda S_1) x_0 = in - S_1 y_1 - \sum_{i=2}^{R+1} \sigma_i y_i
$$

Thus $\sigma_1 = S_1$, gain $\gamma_0 = 1/g = 1 - \lambda S_1$, and

$$
x_0 = g \left(in - \sum_{i=1}^{R+1} \sigma_i y_i \right)
$$

When x_0 is obtained, the non-recursive part of the filter and updating of unit delay states do not cause any realization problems. Based on this transform from α_i to σ_i we may now write an efficient algorithm for the computation of the mapping

$$
\begin{aligned}
&\sigma_{R+1} = \lambda \alpha_R; \quad S_R = \alpha_R; \\
&\text{for } i = R, R-1, ..., 2 \\
&\quad S_{i-1} = \alpha_{i-1} - \lambda S_i; \\
&\quad \sigma_i = \lambda S_{i-1} + S_i; \\
&\text{end} \\
&\sigma_1 = S_1; \quad 1/g_0 = \sigma_0 = 1 - \lambda S_1;
\end{aligned}
$$

Equivalently to this, coefficients σ_i, $k = 0, ..., R+1$, may be computed from α_i using explicit formulas:

$$
\begin{aligned}
1/g_0 &= \sigma_0 = 1 + \sum_{i=1}^{R} \alpha_i (-\lambda_i)^i \\
\sigma_1 &= \sum_{i=1}^{R} \alpha_i (-\lambda_i)^{i-1}, and \\
\sigma_k &= \sum_{i=k}^{R} \alpha_i (-\lambda_i)^{i-k} - \sum_{i=k-1}^{R} \alpha_i (-\lambda_i)^{i-k+2}
\end{aligned}
$$

The method of making warped recursive filters realizable by moving feedback taps to the outputs of unit delays is a general and systematic approach that can be applied to other warped recursive structures as well. Another example of this technique, applied to warped recursive lattice filters, has been shortly discussed in [15].

4. WIIR PERFORMANCE

The added complexity of WFIR and WIIR structures, when compared to ordinary FIRs and IIRs, may look as extra computational cost in the implementation for practical applications. On the other hand, in applications where the warped resolution yields a natural match to desired filter characteristics the order of filter may be reduced substantially. E.g., in warping to the Bark scale, for full audio bandwidth, a reduction of 5 to 1 or even 10 to 1 may be gained which efficiently counteracts the more complex structure.

We have estimated the cost of WFIR and WIIR filters when using popular DSP processors, such as the Motorola 56000 or the TI TMS320C30. WFIR filters typically take 3 to 4 instruction cycles per tap compared to 1 for ordinary FIRs. For WIIR filters 4 to 5 instructions are needed per order compared to 2 for direct form II structures. Thus we may conclude than in best cases WIIR filters may be 2 to 5 times faster than traditional IIR filters for purposes where the warping principle works ideally.

Other performance issues are the robustness and the precision requirements. Digital filters are known to be sensitive to precision and quantization effects if poles and/or zeros are located close to each other or clustered in the z-plane. If the poles and zeros, when mapped to the warped frequency domain, are more apart, the filter becomes less critical. This often happens in warped filters, especially in WIIR structures. Our experience shows that in some applications, e.g., Bark-warped WIIRs of orders higher than 100 to 400 are stable and accurate while equivalent IIR direct form filters cannot be realized for orders higher than 20 to 30 when double precision arithmetics is used [8], [9].

5. DISCUSSION

Although introduced long ago, the principles of warped filters are not widely known and their applications have remained relatively few. Warped filters may be considered as hierarchically structured filters in which the basic building blocks are filters themselves. From the viewpoint of possible transfer functions, warped filters do not yield anything that could not be achieved with traditional filter structures. However, they reveal interesting theoretical aspects as well as advantages when used in practical applications. In this paper we have studied the realization of warped IIR filter structures, a task that is not as straightforward as it may first appear.

The WIIR structures studied in this paper were already found well suited in several audio applications where the human auditory perception is modeled or where the characteristics of a physical system to be modeled follow similar guidelines of frequency resolution. Warped filters are also interesting in cases where the frequency response—for example the cut-off frequency—must be controlled using a single parameter.

In addition to further applications, the authors are working on generalizing the idea of warped filters. One of the possible extensions is to apply more arbitrary warping functions, including logarithmically shaped scales and focusing more resolution to arbitrary parts of the frequency scale.

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